Spatial vector soliton and its collisions in isotropic self-defocusing Kerr media

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(Received 18 May 2006; revised manuscript received 30 January 2007; published 14 June 2007)

A fairly general form of the two-component (dark-dark) vector one-soliton solution of the integrable coupled nonlinear Schrödinger equation (Manakov model) with self-defocusing nonlinearity is obtained by using the Hirota method. It couples two dark components with the same envelope width, envelope speed, and envelope trough location using two complex arbitrary parameters not only in the envelope amplitude but also in the complex modulation. Although it has the freedom to change its pulse width without affecting its speed, it can also tune its grayness (depth of the pulse relative to background) without disturbing the envelope width and speed. The variations in peak power against the depth of localization of two dark components are investigated with and without a parametric restriction. The collision between many dark-dark vector solitons has also been studied by constructing a multisoliton solution with more free parameters.

DOI: 10.1103/PhysRevE.75.066605 PACS number(s): 42.65.Tg, 42.81.Dp, 41.20.Jb

I. INTRODUCTION

The coupled nonlinear Schrödinger (CNLS) equations which describe pulse propagation in birefringent nonlinear fibers have been extensively studied [1–3]. Specifically Menyuk [4] showed numerically that above a certain amplitude threshold two polarizations strongly interact and nonlinear pulses consisting of both polarizations are formed. Thus the Kerr nonlinearity compensates not only group dispersion but also polarization dispersion and forms a steady nonlinear pulse [5,6]. This polarized pulse is called the vector soliton even though it is not obtained as a solution of an integrable system. Such a physically applicable CNLS equation reduces to the integrable Manakov model [7] under certain parametric conditions. Therefore one can derive its vector soliton solutions mathematically. The vector soliton solutions of this model can be spatial or temporal and form using two orthogonally polarized components of a single-input vector field. Intense theoretical and experimental investigations show that the Manakov model provides a theoretical framework for understanding certain experimental observations [2,3,8,9]. Very recently, Anastassiou *et al.* [10] observed through experiments the transfer of information from one vector spatial soliton of the Manakov model to another via collision with a third intermediate soliton. This energy exchange collision was, however, first proposed theoretically by constructing a multisoliton solution of the Manakov model [11]. It is a well-known fact that the two-component Manakov model possesses the Painlevé integrability property and is a completely integrable soliton system [12]. Further integrable cases of multicomponent CNLS systems satisfying the Painlevé property were identified, and associated Bäcklund transformations and Hirota bilinear forms to obtain soliton solutions were systematically derived [13]. Soliton solutions for different kinds of CNLS equations, including mixed types, have been intensively studied in recent times [14–18]. Many mixed type N-CNLS systems having the Painlevé property were also found in Ref. [13].

Generally speaking, the Manakov model supports different classes of vector soliton pairings: namely, bright-bright [7,11,16], dark-dark [17,18], and bright-dark [17,18] pairings. They represent two spatial (or temporal) solitons whose shapes do not change during propagation. Moreover, they are helpful in finding the answer to the question, what will happen to the solitary waves of different polarization modes when they get coupled through cross-phase modulation? In this connection three decades ago Manakov derived the bright-bright temporal vector soliton solution by using the inverse scattering transform (IST) method [7] to describe the interaction between two orthogonally polarized components of an input vector field. Very recently dark-dark and bright-dark vector solitons were also derived by using the IST method [18]. However, nearly a decade ago, dark-dark [16] and bright-dark [17] vector soliton solutions were realized with more arbitrary parameters by using Hirota's method.

A seven-parameter (real) dark-dark soliton solution was obtained in [16] by solving the Manakov model using the Hirota method. The role of these seven parameters were characterized [17] and were used to describe the spatial vector soliton in isotropic self-defocusing media. From this characterization it is obvious that, unlike the case of bright-bright vector solitons [7,11] with six real free parameters for the polarization vector, pulse width, and speed, the dark-dark soliton solution with seven arbitrary parameters is not sufficient to account for the polarization vector with independent free parameters required to define the pulse width and speed of an input vector field. In this paper we show that this problem is due to insufficient arbitrary parameters in the dark-dark vector one-soliton solution [16–18]. We have arrived at this conclusion by constructing dark-dark vector solitons with eight arbitrary parameters using the Hirota method.

The dark-dark vector soliton solution obtained with eight free real parameters couples two different dark components with the same envelope width, envelope speed, and envelope trough location by using two complex arbitrary parameters not only in the amplitude part of dark components but also in their complex modulation in the longitudinal coordinate. The general solution has the freedom to change the depth of localization of each component without affecting both the pulse width and speed. Similarly it is also possible to change

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the pulse width of the general solution without affecting the speed or vice versa. The total peak power of this dark-dark vector soliton solution depends on all of the eight free parameters. However, if we assume the depth of localization of the two dark components to be equal, then the total peak power becomes constant against the coupling parameters. Therefore, in this case, by using the coupling parameters as a polarization vector we can distribute the total peak power among the components of the eight-parameter dark-dark vector soliton. It is also interesting to note that under a parametric condition the total peak power of dark vector solitons depends only on envelope width as in the case of bright-bright vector solitons. In this case the coupling parameters can be admitted to form components of the polarization vector whether the dark-dark vector soliton's components have the same or different depth of localization. Moreover, our general solution under a parametric condition has the freedom to vary its speed without affecting the amplitude and pulse width. It is also possible to tune the depth of localization without the affecting amplitude, pulse width, and speed of soliton. However, for any parametric choice, complex modulation in the longitudinal coordinate varies with all eight free parameters.

The proof for the above statements is obtained in Sec. II by constructing a more general dark-dark vector one-soliton solution of the Manakov model. In Sec. III, we have a generalized dark-dark multisoliton solution of the Manakov model to describe the collision between different dark-dark vector one-soliton solutions having the same background field and enhanced number of arbitrary parameters.

It is to be mentioned that this present study has been extended to construct bright-dark spatial vector solitons with seven arbitrary parameters. Unlike the bright-bright and dark-dark vector soliton solutions of the Manakov model, a singular point of our generalized bright-dark pair generated by us separates the existing bright-dark region into two parts: namely, self-focusing and self-defocusing regions. Further we have also constructed a bright-dark multisoliton solution that enables us to eliminate the beating and breathing effects in bright-dark collisions with the help of additional parameters in colliding solitons. It is beyond the scope of this paper to report this, due to the complicated mathematics involved with the singular point.

II. DARK-DARK SPATIAL VECTOR ONE-SOLITON SOLUTION OF THE MANAKOV MODEL AND ROLE OF ITS EIGHT ARBITRARY PARAMETERS

The CNLS equations derived to describe the $(1+1)$-dimensional propagation of high-intensity light of arbitrary polarization in isotropic self-defocusing media [1] can be reduced to the Manakov model

$$i\frac{\partial u_m}{\partial z}+\frac{\partial^2 u_m}{\partial x^2}-2(|u_m|^2+|u_{3-m}|^2)u_m=0, \quad m=1,2, \quad (1)$$

if the ratio between nonlinearity due to self-phase modulation (SPM) and cross-phase modulation (CPM) is unity and the four-wave mixing (FWM) term is zero. The FWM term can be averaged out to zero [4] if the effects related to the birefringence are stronger than the nonlinear effects. In real materials the SPM/CPM ratio can take on a wide range of values. In most cases it is positive. For example, in $Al_xGa_{1-x}As$ crystal at frequencies near half the band gap, this ratio is close to unity [19]. The case SPM/CPM$=1$ is a special one. The wave propagation is then described by the above integrable Manakov equations. Here $u_1(z,x)$ and $u_2(z,x)$ are two orthogonally polarized components of a slowly varying envelope pulse $\mathbf{u}$; z and x are the longitudinal and transverse coordinates. The above equation (1) can be decoupled into two NLS equations. In the case of self-defocusing Kerr nonlinearity, the NLS equation supports a dark-soliton solution which can be classified into fundamental dark (FD) and gray dark (GD) based on its depth of localization. When two dark soliton solutions of NLS equation couple as dictated by Eq. (1), then one can realize different dark-dark pairs from a general vector dark soliton of Eq. (1) (namely, GD-GD pair with equal or unequal depth components, FD-FD and FD-GD/GD-FD pairs). When a dark-dark pair converts its form by changing its depth of localization how can power and polarization vector of a general dark-dark soliton solution vary against parameters for the depth of localization of two dark components? In order to answer this question, we have analyzed the coupling between u_1 and u_2 with same envelope width, envelope speed, and envelope trough location by constructing a more general eight-parameter dark-dark vector soliton with the help of the Hirota technique as described below.

A. Dark-dark one-soliton solution

By using the Hirota bilinear transformations [16] $u_1=g/f$ and $u_2=h/f$, in which $g(z,x)$ and $h(z,x)$ are complex functions and $f(z,x)$ is a real function, the Manakov equation can be rewritten as a set of homogeneous equations involving only quadratic terms (f^2,fg,g^2) acted on by the Hirota bilinear operator $D_z^m D_x^n(g\cdot f)=(\partial_z-\partial_{z'})^m(\partial_x-\partial_{x'})^n[g(z,x)f(z',x')]|_{z=z',x=x'}$:

$$f[(iD_z+D_x^2)g\cdot f]-g[D_x^2 f\cdot f+2(g\cdot g^*+h\cdot h^*)]=0,$$

$$f[(iD_z+D_x^2)h\cdot f]-h[D_x^2 f\cdot f+2(g\cdot g^*+h\cdot h^*)]=0. \quad (2)$$

Without loss of generality one can decouple the above equation into three simple equations by introducing an unknown decoupling constant λ as

$$\mathcal{B}_1 g\cdot f=0, \quad \mathcal{B}_1 h\cdot f=0, \quad \mathcal{B}_2 f\cdot f=-2(g\cdot g^*+h\cdot h^*), \quad (3)$$

where the bilinear operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined as $\mathcal{B}_1=(iD_z+D_x^2-\lambda)$ and $\mathcal{B}_2=(D_x^2-\lambda)$.

The next step is to use a formal power series quadratic ansatz for the functions $g=g_0(1+\chi^2 g_2)$, $h=h_0(1+\chi^2 h_2)$, and $f=1+\chi^2 f_2$ in Eqs. (3) and to look for g_0, g_2, h_0, h_2, and f_2 by solving equations collected with the same power of χ. There are many ways to define g_0, g_2, h_0, h_2, and f_2. But the Hirota method needs a judicious ansatz for input expressions g_0 and h_0 (corresponding to equation of χ^0) to provide practically interesting nontrivial solutions or solutions with more parameters. For example, in Ref. [11], the energy exchange between the components of the colliding bright-bright vector solitons was noted recently just by modifying the earlier ansatz used for the input functions in the power series [16].

There exist more open questions than solved problems in the theory of multicomponent dark-dark solitons of the integrable Manakov model [16–18] due to the lack of exact dark-dark soliton solutions with enough degrees of freedom as mentioned in Sec. I. The loss of parameters in the dark-dark vector soliton solution obtained from the Hirota method is possible [16] if the ansatz used for the input functions g_0 and h_0 is not perfect. So we have started our further analysis with the assumptions for g_0 and h_0 as $\tau_1 R^{-1/2}\exp(i\psi_1)$ and $\tau_2 R^{-1/2}\exp(i\psi_2)$, respectively, where τ_1 and τ_2 are complex parameters and $\psi_j = l_j x - (l_j^2+\lambda)z + \psi_j^{(0)}$, $j=1,2$, in which l_j and $\psi_j^{(0)}$ are real parameters. With this assumption, if one follows the usual algorithm used in [16] to derive the dark-dark pair, an expression for R arises which is real and helpful to define the polarization vector by playing a role in the amplitude part of the resultant solution. In an earlier investigation R is not used in g_0 and h_0 and consequently a dark-dark pair with seven parameters is obtained. By including the parameter R as mentioned before we obtain a more general dark-dark pair with eight free parameters as

$$u_j = P_j e^{i\delta_j z} e^{i(\psi_j'+\alpha_j)} e^{2iA_j^2 z} A_j \{ i \sin\alpha_j + \cos\alpha_j \tanh[(\eta_1 + \Gamma)/2]\}, \quad j=1,2, \tag{4}$$

where $P_1 = \frac{|\tau_1|}{|cg|}\Delta^{-1/2}$, $P_2 = \frac{|\tau_2|}{|ch|}\Delta^{-1/2}$, $A_1 = \frac{|cg|}{2k_1}$, $A_2 = \frac{|ch|}{2k_1}$, $\delta_1 = \frac{|\tau_1|^2+|\tau_2|^2-|cg|^2\Delta}{2k_1^2\Delta}$, and $\delta_2 = \frac{|\tau_1|^2+|\tau_2|^2-|ch|^2\Delta}{2k_1^2\Delta}$ in which $cg = k_1^2 - i(2l_1k_1+\omega_1)$, $ch = k_1^2 - i(2l_2k_1+\omega_1)$, $\Delta = \left(\frac{|\tau_1|^2}{|cg|^2} + \frac{|\tau_2|^2}{|ch|^2}\right)$, $\alpha_1 = \arctan(cg_I/cg_R)$, $\alpha_2 = \arctan(ch_I/ch_R)$, $\eta_1 = k_1 x + \omega_1 z$, $\psi_j' = l_j x - l_j^2 z + \psi_j^{(0)}$, and $\Gamma = \eta_1^{(0)} + \ln[4k_1^2\Delta]$.

In the above solution, l_1, l_2, k_1, and ω_1 are four real arbitrary parameters, $\tau_1 = \tau_{1R} + i\tau_{1I}$ and $\tau_2 = \tau_{2R} + i\tau_{2I}$ are two free complex parameters, and the suffixes R and I represent the real and imaginary parts, respectively. These eight arbitrary real parameters l_1, l_2, k_1, ω_1, τ_{1R}, τ_{1I}, τ_{2R}, and τ_{2I} can be used to characterize spatial vector solitons in the isotropic self-defocusing Kerr media. For example, if $l_1 = l_2 = \frac{-\omega_1}{2k_1}$, then $\alpha_1 = \alpha_2 = 0$. Consequently Eq. (4) has two FD components. Thus the parameter l_1 (l_2) can be used to define the depth of localization of a dark component u_1 (u_2) of a single-input pulse with definite envelope width and envelope speed decided, respectively, by using k_1 and ω_1. It is also obvious to note from Eq. (4) that the free parameter k_1 is helpful to define envelope width without disturbing the envelope speed while ω_1 is useful to tune the envelope speed without affecting the envelope width. Hence the two components of dark-dark pair (4) have the same envelope width and envelope speed even though each component of Eq. (4) has separate parameters to define its grayness. The remaining four free real parameters τ_{1R}, τ_{1I}, τ_{2R}, and τ_{2I} in P_j can be used to characterize the polarization vector of an input light pulse or intensity of each component as described in the following subsections.

By using the expressions for α_1 and α_2, one can read the parameter k_1 used to define the widths of u_1 and u_2 as $k_1 = 2A_1\cos\alpha_1 = 2A_2\cos\alpha_2$ (equal widths). Similarly ω_1 for u_1 and u_2 can be read as $\omega_1 = -4A_1\cos\alpha_1(A_1\sin\alpha_1 + l_1) = -4A_2\cos\alpha_2(A_2\sin\alpha_2 + l_2)$ (equal speeds). The term A_j can be used to calculate the peak power as mentioned below. The central position of a soliton at any z can be calculated by solving the equation $\eta_1 + \Gamma = 0$ for x. Therefore the real constant $\eta_1^{(0)}$ in this equation is not arbitrary and can be scaled by z or Γ. Similarly the complex parameters $\psi_j^{(0)}$ can also be absorbed by the free parameters τ_j. Anyhow Γ contributes to the phase shift due to dark-dark vector soliton collisions as shown in the next section by constructing a multisoliton solution of the Manakov model (1).

We can decouple the above solution by setting $\tau_1 = 0$ or $\tau_2 = 0$. If $\tau_2 = 0$, Eq. (4) becomes

$$\tilde{u}_1 = e^{i(\psi_1'+\alpha_1)} e^{2iA_1^2 z} A_1\{ i\sin\alpha_1 + \cos\alpha_1 \tanh[(\eta_1+\Gamma)/2]\}. \tag{5}$$

It can be easily checked that Eq. (5) is a solution of the scalar NLS equation. It has five real free parameters l_1, k_1, ω_1, and imaginary and real parts of $\psi_1^{(0)}$, respectively for grayness, pulse width, speed, amplitude, and complex phase. Similarly for $\tau_1 = 0$, Eq. (4) becomes

$$\tilde{u}_2 = e^{i(\psi_2'+\alpha_2)} e^{2iA_2^2 z} A_2\{ i\sin\alpha_2 + \cos\alpha_2 \tanh[(\eta_1+\Gamma)/2]\}. \tag{6}$$

Here l_2, k_1, ω_1, and $\psi_2^{(0)}$ are free parameters. With the help of Hirota's method we have coupled Eqs. (5) and (6) with the same speed, pulse width, and envelope trough location by using Eq. (1) as shown above. Therefore the resultant coupled dark soliton has eight free parameters and two components with the same pulse width and speed as expected. During this coupling, the complex modulation in z of Eqs. (5) and (6) is, respectively, shifted by δ_1 and δ_2. Even though our general solution (4) has a complicated form, we can characterize different dark-dark pairs associated with it by using the parameters l_1 and l_2 as given below.

B. Relation between polarization vector, peak power, and depth of localization

From the general solution (4) one can easily note that the depth of localization of u_j changes as α_j varies with respect to the parametric values of k_1, ω_1, and l_j. For instance, if α_j becomes zero, Eq. (4) represents an FD-FD pair. In addition the variations in α_1 and α_2 against k_1 and ω_1 are always same for a given l_j. It means that by tuning k_1 and ω_1 we can not only change the pulse width and speed of an input light pulse, respectively, but can also simultaneously modify the depth of localization of its two dark components by an equal amount. Anyhow, the depths of localization of two dark polarization components are the same or different, only if $\alpha_1 = \alpha_2$ or $\alpha_1 \neq \alpha_2$ (i.e., $l_1 = l_2$ or $l_1 \neq l_2$). Thus as mentioned before the only choice to tune the depth of the jth component without affecting other component's depth is l_j. Therefore by using this sensitive parameter l_j, we can realize the following dark-dark pairs.

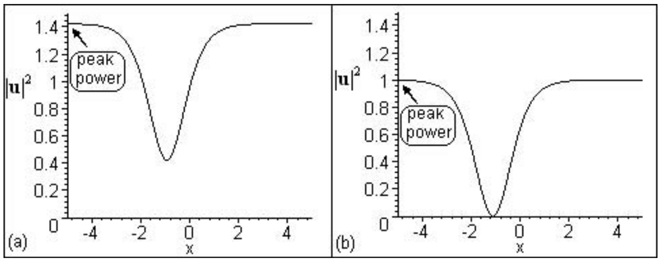

FIG. 1. The changes in peak power of the GD-GD pair with equal-depth components when it converts into the FD-FD pair by taking $\alpha_1=\alpha_2=0$. (a) GD-GD pair with equal depth and (b) FD-FD pair.

Under a parametric choice $l_1 \neq l_2$, the solution (4) can be classified into three well-known forms with respect to depth of localization of u_j as described below.

(i) GD-GD pair with unequal depth components; it can be characterized by using the general solution (4), with $l_1 \neq l_2 \neq \frac{-\omega_1}{2k_1}$.

(ii) FD-GD pair, which appears from Eq. (4) if $l_1 = \frac{-\omega_1}{2k_1} \neq l_2$. For this choice, $\alpha_1 = 0$ and consequently u_1 takes the FD form.

(iii) Similarly the special case GD-FD pair arises if $l_2 = \frac{-\omega_1}{2k_1} \neq l_1$.

The role of other arbitrary complex parameters τ_1 and τ_2 associated with Eq. (4) can be explained by deriving the following relation from Eq. (4) as

$$(|u_1|^2 + |u_2|^2) = (4k_1^2\Delta)^{-1}\{(|\tau_1|^2 + |\tau_2|^2) - (|\tau_1|^2\cos^2\alpha_1 + |\tau_2|^2\cos^2\alpha_2)\sec h^2[(\eta_1 + \Gamma)/2]\}. \quad (7)$$

By using Eq. (7) one can easily check that the maximum value of Eq. (7) is proportional to $(4k_1^2\Delta)^{-1}$. It implies that the peak power of the vector soliton pulse **u** (sum of the peak power of u_1 and u_2) can vary against all soliton parameters including τ_1 and τ_2 only if $\alpha_1 \neq \alpha_2$. Such a variation against τ_1 and τ_2 disappears if $\alpha_1 = \alpha_2$ as explained below. Therefore one can argue by using Eq. (7) that the peak powers of different dark-dark pairs associated with Eq. (7) are different. That is while the dark-dark pair changes its form by varying its depth of localization (using l_j) its peak power also changes.

In the case of $\alpha_1 = \alpha_2$ which implies $l_1 = l_2$, Eq. (4) can be used to represent two different dark-dark pairs: namely,

(a) GD-GD pair with equal depth components $\left(\alpha_1 = \alpha_2 \neq 0 \Rightarrow l_1 = l_2 \neq \frac{-\omega_1}{2k_1}\right)$ and

(b) FD-FD pair if $\alpha_1 = \alpha_2 = 0$ (i.e., $l_1 = l_2 = \frac{-\omega_1}{2k_1}$). This case was observed experimentally in SBN crystal [9] and has also been characterized to describe the propagation of two incoherent optical beams having the same wavelength and polarization in biased photovoltaic photorefractive crystals [9,20].

For $\alpha_1 = \alpha_2$ one can derive maximum value of $|\mathbf{u}|^2$ (i.e., peak power of the **u**) of the GD-GD pair with equal depth components as $|cg|^2/(4k_1^2)$ by using Eq. (7). Similarly with $\alpha_1 = \alpha_2 = 0$ one can see the peak power of the FD-FD pair from Eq. (7) as $k_1^2/4$. From these expressions one can understand that the peak power of FD-FD and GD-GD pairs are independent of τ_1 and τ_2. However, the peak power of one pair differs from another (i.e., peak power of FD-FD depends on k_1 while that of GD-GD is defined by k_1, ω_1, and l_1). This is confirmed by the numerical plots of Eqs. (4) and (7), shown in Figs. 1 and 2. The GD- GD pair in Fig. 1(a) for the parametric choices $|\tau_1| = 1.0$, $|\tau_2| = 0.5$, $\alpha_1 = \alpha_2 = 0.57$, $k_1 = 2.0$, and $\omega_1 = 1.0$ becomes the FD-FD pair as shown in Fig. 1(b) if both α_1 and α_2 change its value to zero. During this transition we find from Figs. 1 and 2 that the peak power of the GD-GD pair varies from $|cg|^2/(4k_1^2)$ to $k_1^2/4$ as one expects from the earlier discussion. Anyhow the peak power of these vector solitons can be distributed among its components by using τ_j as shown in Fig. 2. This distribution can be modeled by using the mathematical expression $P_j = \tau_j/(\sqrt{|\tau_1|^2 + |\tau_2|^2})$ under the condition $\alpha_1 = \alpha_2$ as $|P_1|^2 + |P_2|^2 = 1$. One can note that such a kind of peak power distribution could not appear against τ_j in the case of dark-dark pairs with $\alpha_1 \neq \alpha_2$. Because if $\alpha_1 \neq \alpha_2$, the peak power of **u** varies with τ_1 and τ_2 as shown by Eq. (7). For example this is shown in Fig. 3(a) against τ_2 by taking $\alpha_1 \neq \alpha_2$. Therefore by tuning τ_1 and τ_2 one can vary the power of the polarization component randomly if $\alpha_1 \neq \alpha_2$ even though $|P_1|^2 + |P_2|^2 = 1$. However, one can bypass such a variation in this case with the help of a parametric condition as shown in the following subsection.

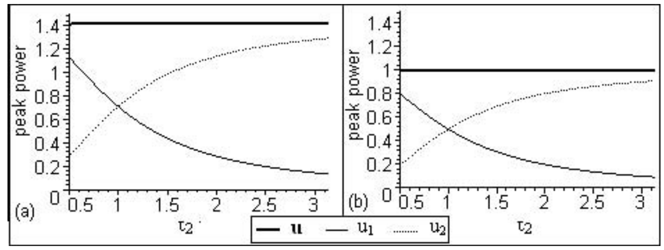

FIG. 2. The sum of power distributed among the polarization components of vector field in Fig. 1 is conserved against τ_2 if $\alpha_1 = \alpha_2$.

C. Polarization vector of all dark-dark pairs with seven free parameters

For $4k_1^2\Delta = 1$, the general solution (4) with eight free parameters reduces its form to the dark-dark solution reported

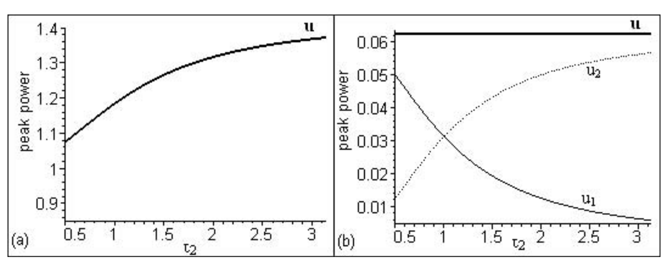

FIG. 3. (a) reflects the fact that the peak power of Eq. (7) is not constant against τ_2 if $\alpha_1 = -0.14$ and $\alpha_2 = 0.57$ (note that although $\alpha_1 \neq \alpha_2$, other parametric values are as in Fig. 1). (b) shows that the power variation in (a) disappears under a parametric condition, Eq. (8).

with seven arbitrary parameters [16] by using the Hirota method. Specifically under this condition one can read the amplitude part of earlier solution from Eq. (4) as $\tau_j/2$. This shows that each polarization component of the earlier solution [16] has a single parameter in its amplitude part and sets restrictions on characterization of the polarization vector and power of different dark-dark pairs. In addition it makes one [17] use the constraint $4k_1^2\Delta=1$ to define the pulse width in terms of speed. But the solution (4) has eight arbitrary real parameters without any restrictions. Therefore with the help of Eq. (4), one can easily write Eq. (7) by introducing a condition

$$\left[\frac{\cos^2\theta}{|cg|^2}+\frac{\sin^2\theta}{|ch|^2}\right]^{-1}=1, \tag{8}$$

as

$$|u_1|^2+|u_2|^2=\frac{1}{4k_1^2}\{1-(\cos^2\theta\cos^2\alpha_1+\sin^2\theta\cos^2\alpha_2)\times\sec h^2[(\eta_1+\Gamma)/2]\}, \tag{9}$$

where $\theta=\arccos(|\tau_1|/\sqrt{|\tau_1|^2+|\tau_2|^2})=\arcsin(|\tau_2|/\sqrt{|\tau_1|^2+|\tau_2|^2})$. Equations (7) and (9) reveal that total peak power of Eq. (4) is independent of τ_1 and τ_2 whether $\alpha_1=\alpha_2$ or $\alpha_1\neq\alpha_2$ under the condition (8). Therefore the peak power or maximum value of Eq. (9) does not vary against τ_1 and τ_2 when the dark-dark pair (4) with constraint (8) converts its form by changing its grayness as shown in Fig. 3(b). It is also interesting to note from Eq. (9) that under a parametric condition (8) the total peak power of the dark vector soliton (4) depends only on the envelope width as in the case of bright vector solitons [7].

III. DARK-DARK MULTISOLITON OF THE MANAKOV MODEL

One can study the effect of collisions between vector solitons by comparing their form in the asymptotic limits: namely, $z\to-\infty$ (before collision) and $z\to\infty$ (after collision) of the multisoliton solution. By applying such asymptotic analysis on the dark-dark multisoliton solution of the Manakov model reported by using the Hirota method [16], one can realize that it suffers from the lack of a sufficient number of parameters. Specifically we have noted that the depths of localization of two dark components of each colliding vector soliton are same (i.e., $l_1=l_2$ case only) and the speed of each colliding vector soliton is explicitly defined in terms of the pulse width. But we have obtained a more general dark-dark multisoliton solution which includes a set of parameters (l_1,l_2) to define the depth of localization of two polarization components on background plane waves $[\tau_1(R)^{-1/2}e^{i\psi_1}, \tau_2(R)^{-1/2}e^{i\psi_2}]$ of all colliding vector dark one soliton solutions. These colliding vector dark one soliton solutions have separate parameters k_j and ω_j ($j=1,2,\ldots,N$, for N-soliton solutions) to define their pulse width and speed, respectively. As the Hirota method restricts one to take same background plane waves for all colliding solitons, we have obtained two sets of parameters (l_1,l_2) and (τ_1,τ_2) to define the depth of two polarized components and polarization of all colliding solutions, respectively, with additional restrictions as shown below by constructing dark-dark multisoliton solutions.

A. Two-soliton solution

By following the systematic steps of the Hirota method used to derive multisoliton solutions [16], we first collect a group of equations with same power of χ in the power series expansion for the functions g, h, and f (in the bilinear transformations $u_1=g/f$ and $u_2=h/f$) as $g=g_0(1+\chi^2g_2+\chi^4g_4)$, $h=h_0(1+\chi^2h_2+\chi^4h_4)$, and $f=(1+\chi^2f_2+\chi^4f_4)$ in Eq. (3). Next we look for g_2, h_2, f_2, g_4, h_4, and f_4 by solving the equations collected with same power of χ. In this connection one can show that $g_0=\tau_1(R)^{-1/2}e^{i\psi_1}$ and $h_0=\tau_2(R)^{-1/2}e^{i\psi_2}$ (as in the one-soliton derivation) are solutions of equations with χ^0. After using these plane-wave solutions for g_0 and h_0 in the equations with χ^j, $j=1,2,\ldots,4$, we look for g_2, h_2, f_2, g_4, h_4, and f_4.

Therefore we have named g_0 and h_0 as input functions for the above manipulation and have taken care to define them by considering their role in the required solution. In our case, g_0 and h_0 are aimed at defining background plane waves where the polarization component of each colliding soliton localizes its form with different depths. In addition the expression arising for R can be used to define the power and polarization vector of all colliding solitons. But unfortunately the Hirota method is unable to generalize g_0 and h_0 so as to include different background waves for different colliding vector solitons. Therefore by having a single-plane-wave solution for g_0 and h_0 as in the one-soliton case (to support the same background plane waves of all colliding solitons with the same polarization) we have obtained a dark-dark two-soliton solution with the help of Hirota's algorithm for multisoliton solutions as

$$u_j=\frac{\tau_je^{i\psi_j+\zeta_+}[(\sqrt{A_{12}}/R)\cosh(\xi_1+\zeta_+)-\cosh(\xi_2+\zeta_-)]}{\sqrt{(A_{12}/R)}\cosh(\xi_1)+\sqrt{R}\cosh(\xi_2)}, \tag{10}$$

where $\xi_1=[\eta_1+\eta_2+\ln(A_{12})]/2$, $\xi_2=(\eta_1-\eta_2)/2$, $\zeta_\pm=(c_{j1}\pm c_{j2})/2$, $c_{1j}=\ln\left(\frac{cg_j}{cg_j^*}\right)$, and $c_{2j}=\ln\left(\frac{ch_j}{ch_j^*}\right)$, in which $cg_j=k_j^2-i(2l_1k_j+\omega_j)$, $ch_j=k_j^2-i(2l_2k_j+\omega_j)$, and $\eta_j=k_jx+\omega_jz$, where $j=1,2$. Here $A_{12}=\frac{R^2(k_1^2k_2^2(k_1-k_2)^2+(k_2\omega_1-k_1\omega_2)^2)}{k_1^2k_2^2(k_1+k_2)^2+(k_2\omega_1-k_1\omega_2)^2}$ and $\lambda=2(|\tau_1|^2+|\tau_2|^2)/R$ in which

$$R=4\left(\frac{|\tau_1|^2}{|cg_1|^2}+\frac{|\tau_2|^2}{|ch_1|^2}\right)k_1^2=4\left(\frac{|\tau_1|^2}{|cg_2|^2}+\frac{|\tau_2|^2}{|ch_2|^2}\right)k_2^2. \tag{11}$$

The above solution has a complicated mathematical form and is difficult to understand. However, one important advantage of Hirota's method is that the solution allows easy analysis of the asymptotic behavior by taking a given $\eta_i\to\pm\infty$ in the above two-soliton solution and comparing the result with the one-soliton solution (4). The interpretation of the result in terms of the actual motion of the solitons depends on the sign of k_i and ω_i. In general $n_i=k_i\left(x+\frac{\omega_i}{k_i}z\right)$ so that soliton j is located in the vicinity of the line $x=\frac{-\omega_j}{k_j}z$. Let us change to the frame moving along with soliton j (coordinatized by ϑ) by putting $x=\vartheta-\frac{\omega_j}{k_j}z$. Then $\eta_j=k_j\vartheta$ while for the other soliton m we get $n_m=k_m\Big[\vartheta+\Big(\frac{\omega_m k_j-\omega_j k_m}{k_j k_m}\Big)z\Big]$. Thus if we have for example $k_i>0$ and $\omega_2>\omega_1$, we find that $\eta_j\to\pm\infty$ corresponds to $z\to\pm\infty$. So we obtain the following results.

1. Limit: $z\to-\infty$

Soliton 1:

$$u_j^{1-}=P_j^1A_j^1\{i\sin\alpha_j^1+\cos\alpha_j^1\tanh[(\eta_1+\Gamma^{1-})/2]\}\exp[i\psi_j'+i\alpha_j^1+i\delta_j^1z+2i(A_j^1)^2z],\quad j=1,2. \tag{12}$$

Soliton 2:

$$u_j^{2-}=P_j^2A_j^2\{i\sin\alpha_j^2+\cos\alpha_j^2\tanh[(\eta_2+\Gamma^{2-})/2]\}\exp[i\psi_j'+i(\alpha_j^2-2\alpha_j^1)+i\delta_j^2z+2i(A_j^2)^2z],\quad j=1,2. \tag{13}$$

2. Limit: $z\to+\infty$

Soliton 1:

$$u_j^{1+}=P_j^1A_j^1\{i\sin\alpha_j^1+\cos\alpha_j^1\tanh[(\eta_1+\Gamma^{1+})/2]\}\exp[i\psi_j'+i(\alpha_j^1-2\alpha_j^2)+i\delta_j^1z+2i(A_j^1)^2z],\quad j=1,2. \tag{14}$$

Soliton 2:

$$u_j^{2+}=P_j^2A_j^2\{i\sin\alpha_j^2+\cos\alpha_j^2\tanh[(\eta_2+\Gamma^{2+})/2]\}\exp(i\psi_j'+i\alpha_j^2+i\delta_j^2z+2i(A_j^2)^2z,\quad j=1,2, \tag{15}$$

where $P_1^j=\frac{|\tau_1|}{|cg_j|}\Delta_j^{-1/2}$, $P_2^j=\frac{|\tau_2|}{|ch_j|}\Delta_j^{-1/2}$, $A_1^j=\frac{|cg_j|}{2k_j}$, $A_2^j=\frac{|ch_j|}{2k_j}$, $\alpha_1^j=\arctan\Big(\frac{cg_{jI}}{cg_{jR}}\Big)$, $\alpha_2^j=\arctan\Big(\frac{ch_{jI}}{ch_{jR}}\Big)$, $\delta_1^j=\frac{|\tau_1|^2+|\tau_2|^2-|cg_j|^2\Delta_j}{2k_j^2\Delta_j}$, and $\delta_2^j=\frac{|\tau_1|^2+|\tau_2|^2-|ch_j|^2\Delta_j}{2k_1^2\Delta_j}$ in which $\Delta_j=\Big(\frac{|\tau_1|^2}{|cg_j|^2}+\frac{|\tau_2|^2}{|ch_j|^2}\Big)$, $\Gamma^{1-}=\Gamma^{2+}=\ln(R)$, and $\Gamma^{1+}=\Gamma^{2-}=\ln(A_{12}/R)$.

From the above asymptotic equations, it is obvious that soliton 1 and soliton 2 corresponding to Eqs. (12) and (13) in the limit $z\to-\infty$ are modified to Eqs. (14) and (15) after the collision during their propagation from $z\to-\infty$ to $z\to+\infty$. Further by comparing Eqs. (12)–(15) one can realize that even though there is no energy distribution in the collision between two dark-dark solitons with the same (parallel) polarization as in the bright-bright case [16], the phase shift (Γ^{1-},Γ^{1+} for soliton 1 and Γ^{2-},Γ^{2+} for soliton 2) due to the collision depends on the soliton parameters. Thus our solution is useful to vary the phase shift due to a collision by changing the soliton parameters as shown in a recent experiment [21].

In addition one can note that even though colliding solitons are in the form of Eq. (4), they admit two different relations for same R. However, the expression for A_{12} and the phase shift are also obtained in terms of R without assigning any relation for R. Therefore one can easily check that for $R=1$ and $l_1=l_2$, the above solution reduces to the dark-dark two-soliton solution reported from the Hirota method [16]. Although our solution includes the possibility of $l_1\neq l_2$, two different relations for R are obtained due to the same background field for all colliding solitons. One can interpret this constraint relation (11) in different ways to obtain the polarization vector and power of colliding solitons with the same polarization and background plane waves. For example, we see the following two cases.

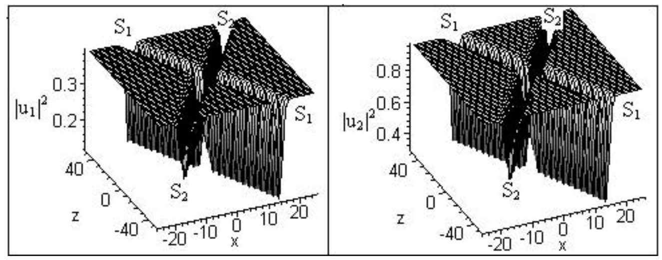

FIG. 4. Head-on collision between S_1 and S_2 under the parametric choices as in the text.

Case I. If we choose $l_1=l_2=l$, then $cg_1=ch_1$, $cg_2=ch_2$, and hence Eq. (11) becomes $\frac{k_1^2}{|cg_1|^2}=\frac{k_2^2}{|cg_2|^2}$. By using this relation we derive

$$l=\frac{(k_1^2-k_2^2)-\left(\frac{\omega_2^2k_1^2-\omega_1^2k_2^2}{k_1^2k_2^2}\right)}{4\left(\frac{\omega_2k_1-\omega_1k_2}{k_1k_2}\right)} \tag{16}$$

and

$$R=4\frac{k_1^2}{|cg_1|^2}(|\tau_1|^2+|\tau_2|^2). \tag{17}$$

Using Eq. (17) one can characterize the colliding solitons as mentioned in Sec. II under the condition $l_1=l_2$.

Case II. To satisfy Eq. (11), we select

$$|\tau_2|^2=\frac{|\tau_1|^2\left(\frac{k_1^2}{|cg_1|^2}-\frac{k_2^2}{|cg_2|^2}\right)}{\left(\frac{k_2^2}{|ch_2|^2}-\frac{k_1^2}{|ch_1|^2}\right)}. \tag{18}$$

Then one can obtain a single relation for R to characterize colliding solitons with $l_1\neq l_2$. In this case one can realize Eqs. (12) and (13) (colliding solitons before collision) with separate free parameters for the pulse width and speed. In

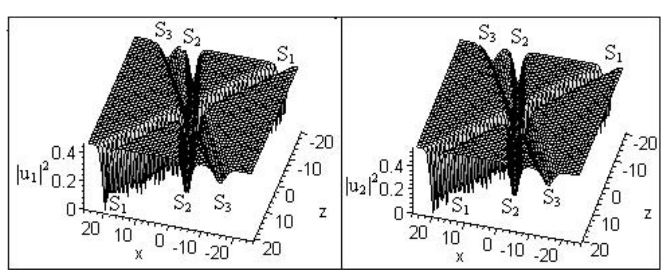

FIG. 5. S_1 makes a head-on collision with S_2 and S_3 while S_2 and S_3 overtake each other under the parametric choices as in the text.

addition these admit a set of parameters (l_1, l_2) to define the depth of localization of two polarization components of each colliding soliton with the same polarization and background field. For example we show collisions between two such dark-dark pairs in Fig. 4 under the parametric choices $|\tau_1| = 1.0$, $k_1 = 1.0$, $k_2 = 1.5$, $\omega_1 = -0.5$, $\omega_2 = 0.5$, $l_1 = -0.5$, and $l_2 = 0.4$.

B. Three-soliton solution

By using the expansion for the functions g, h, and f as $g = g_0(1+\chi^2 g_2 + \chi^4 g_4 + \chi^6 g_6)$, $h = h_0(1+\chi^2 h_2 + \chi^4 h_4 + \chi^6 h_6)$, and $f = (1+\chi^2 f_2 + \chi^4 f_4 + \chi^6 f_6)$ with $g_0 = \tau_1 (R)^{-1/2} e^{i\psi_1}$ and $h_0 = \tau_2 (R)^{-1/2} e^{i\psi_2}$, we have derived a dark-dark three-soliton solution as

$$u_j = \frac{\tau_j e^{i\psi_j + \zeta_+}\{(\sqrt{A_{12}}/R)\cosh(\xi_1 + \zeta_+) - \cosh(\xi_2 + \zeta_-) + e^{\eta_3 + c_{j3}}[\sqrt{(A_{123}/R)}\cosh(\xi_3 + \zeta_+) - (\sqrt{A_{13}A_{23}}/R)\cosh(\xi_4 + \zeta_-)]\}}{\sqrt{(A_{12}/R)}\cosh(\xi_1) + \sqrt{R}\cosh(\xi_2) + e^{\eta_3}[\sqrt{A_{123}}\cosh(\xi_3) + \sqrt{(A_{13}A_{23}/R)}\cosh(\xi_4)]}, \quad (19)$$

where $\xi_1 = [\eta_1 + \eta_2 + \ln(A_{12})]/2$, $\xi_2 = (\eta_1 - \eta_2)/2$, $\xi_3 = [\eta_1 + \eta_2 + \ln(A_{123}/R)]/2$, $\xi_4 = [\eta_1 - \eta_2 + \ln(A_{13}/A_{23})]/2$, $\zeta_\pm = (c_{j1} \pm c_{j2})/2$, $c_{1j} = \ln\left(\frac{cg_j}{cg_j^*}\right)$, and $c_{2j} = \ln\left(\frac{ch_j}{ch_j^*}\right)$, in which $cg_j = k_j^2 - i(2l_1 k_j + \omega_j)$, $ch_j = k_j^2 - i(2l_2 k_j + \omega_j)$, and $\eta_j = k_j x + \omega_j z$, where $j = 1,2,3$. Here $A_{123} = A_{12}A_{13}A_{23}$ in which $A_{ij} = \frac{R^2[k_i^2 k_j^2 (k_i - k_j)^2 + (k_j\omega_i - k_i\omega_j)^2]}{k_i^2 k_j^2 (k_i - k_j)^2 + (k_i\omega_i - k_i\omega_j)^2}$. Here we are having three different relations for R as $R = 4(|\tau_1|^2/|cg_1|^2 + |\tau_2|^2/|ch_1|^2)k_1^2 = 4(|\tau_1|^2/|cg_2|^2 + |\tau_2|^2/|ch_2|^2)k_2^2 = 4(|\tau_1|^2/|cg_3|^2 + |\tau_2|^2/|ch_3|^2)k_3^2$. By solving these equations as shown in the above section, one can use Eq. (19) to define the interaction between different three dark-dark pairs. For example, we show three-soliton collision in Fig. 5 for the parametric choices $|\tau_1| = 1.0$, $|\tau_2| = 1.2$, $k_1 = 2.0$, $k_2 = 1.5$, $k_3 = 1.0$, $l_1 = -0.4$, and $l_2 = 0.5$. The induced phase shift due to collisions between any two solitons of this three-soliton system can be analyzed in terms of the phase shift due to general two-soliton collisions. The same procedure can be extended to generalize an N-soliton solution [16].

IV. CONCLUSION

The dark vector soliton solution of Eq. (1) is obtained by using the Hirota method. We have characterized this solution with obscure structure and nonobvious wrinkle to define its polarization vector, envelope width, envelope speed, envelope amplitude, grayness, and complex modulation. In addition we have investigated the collision between dark-dark pairs by deriving a dark-dark multisoliton solution with more degrees of freedom.

We strongly believe that our results may be of use to develop the theory of bright-dark vector soliton collisions and its applications to optical computations. Much progress has been made along these lines and is to be reported later. Many such investigations were not possible earlier due to insufficient arbitrary parameters in the dark component. Therefore this study may provide a theoretical framework for comprehending future experimental work with isotropic self-defocusing media.

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